Un panorama de l'équation des gaz granulaires
Modélisation, analyse théorique et numérique

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Séminaire de l'équipe Modélisation et Contrôle

Outline of the Talk

Wm Wilson.
15th Dec. 1913.

H W Elphinstone
March 30/59

ON THE

STABILITY OF THE MOTION

OF

SATURN'S RINGS.

AN ESSAY,

WHICH OBTAINED THE ADAMS PRIZE FOR THE YEAR 1856, IN THE UNIVERSITY OF CAMBRIDGE.

BY J. CLERK MAXWELL, M.A.

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.
PROFESSOR OF NATURAL PHILOSOPHY
IN THE MARISCHAL COLLEGE AND UNIVERSITY OF ABERDEEN.

"E pur si muove."

Cambridge:
MACMILLAN AND CO.
AND 23 HENRIETTA STREET, COVENT GARDEN, LONDON.
1859.

in order to be permanent, and that this is inconsistent with its outer and inner parts moving with the same angular velocity. Supposing the ring to be fluid and continuous, we found that it will be necessarily broken up into small portions.

We conclude, therefore, that the rings must consist of disconnected particles; these may be either solid or liquid, but they must be independent. The entire system of rings must therefore consist either of a series of many concentric rings, each moving with its own velocity, and having its own systems of waves, or else of a confused multitude of revolving particles, not arranged in rings, and continually coming into collision with each other.

Taking the first case, we found that in an indefinite number of possible cases the mutual perturbations of two rings, stable in themselves, might mount up in time to a destructive magnitude, and that such cases must continually occur in an extensive system like that of Saturn, the only retarding cause being the possible irregularity of the rings.

The result of long-continued disturbance was found to be the spreading out of the rings in breadth, the outer rings pressing outwards, while the inner rings press inwards.

The final result, therefore, of the mechanical theory is, that the only system of rings which can exist is one composed of an indefinite number of unconnected particles, revolving round the planet with different velocities according to their respective distances. These particles may be arranged in series of narrow rings, or they may move through each other irregularly. In the first case the destruction of the system will be very slow, in the second case it will be more rapid, but there may be a tendency towards an arrangement in narrow rings, which may retard the process.

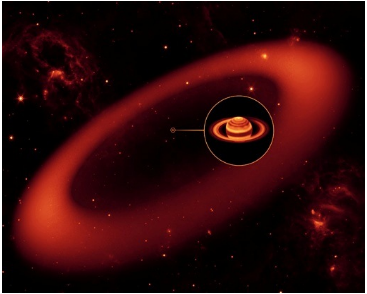
We are not able to ascertain by observation the constitution of the two outer divisions of the system of rings, but the inner ring is certainly transparent, for the limb of Saturn has been observed through it. It is also certain, that though the space occupied by the ring is transparent, it is not through the material parts of it that Saturn was seen, for his limb was observed without distortion; which shows that there was no refraction, and therefore that the rays did not pass through a medium at all, but between the solid or liquid particles of which the ring is composed. Here then we have an optical argument in favour of the theory of independent particles as the material of the rings. The two outer rings may be of the same nature, but not so exceedingly rare that a ray of light can pass through their whole thickness without encountering one of the particles.

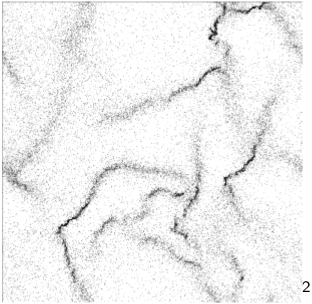
Finally, the two outer rings have been observed for 200 years, and it appears, from the careful analysis of all the observations by M. Struvé, that the second ring is broader than when first observed, and that its inner edge is nearer the planet than formerly. The inner ring also is suspected to be approaching the planet ever since its discovery in 1850. These appearances seem to indicate the same slow progress of the rings towards separation which we found to be

The Granular Gases Equation

The granular gases equation describes the behavior of a dilute gas of particles when the only interactions taken into account are binary inelastic collisions

This concerns various systems, such as pollen dissemination, avalanches, planetary rings, etc.[1], ...




[2]

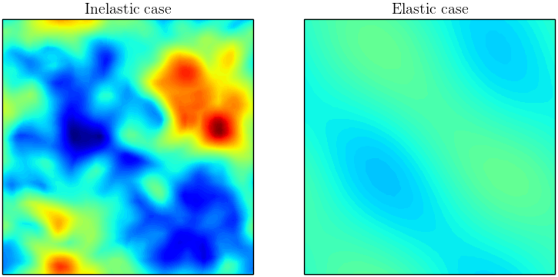
[1] see also Kawai, Shida, *J. Phys. Soc. Japan* (1990)
[2] Courtesy of T. Pöschel

The clustering phenomenon

Numerical method. Rescaling velocity method, using some qualitative properties of the equation[3].

Results. Local density, $\varepsilon = 0.05$: granular gas (left) vs. perfect elastic gas (right), at time $T = 2$.

Numerical parameters. $2d_x \times 3d_v$ model, $N_x = N_y = 100$, $N_{v_x} = N_{v_y} = N_{v_z} = 32$.



→ Consistent with experiments (Brillantov, Pöschel, 2004).

[3]F. Filbet, TR, *JCP* (2013)

A Boltzmann-like Kinetic Equation

General Scaled Equation

Study of a particle distribution function $f^\varepsilon(t,x,v)$, depending on time $t > 0$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, solution to

$$(1) \qquad \begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \dfrac{1}{\varepsilon}\, \mathcal{Q}_e(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon(0,x,v) = f_0(x,v), \end{cases}$$

where $\mathcal{Q}_e$ is the inelastic collision operator, describing the microscopic collision dynamic and ε is a scaling parameter.

- $\rightarrow$ ε is the Knudsen number, ratio of the mean free path between collision by the typical length scale of the problem;
- $\rightarrow$ $\mathcal{Q}_e$ only acts on the v variable;
- $\rightarrow$ Boundary conditions in space needed to describe completely the problem.

Modeling Assumptions

Microscopic dynamics

- Localized interactions: particles interact only by contact (collision), at a given time t and a given position x;
- Diluted gases: collisions occur between two particles at the same time (we neglect the collisions of three particles or more);
- ~~Micro-reversible collisions: the collision dynamics is time-reversible (at the microscopic level)~~;
- Boltzmann chaos assumption: the velocity of two colliding particles are uncorrelated before collision.

The microscopic collision process is said to be

$\rightarrow$ elastic when the kinetic energy is conserved during a collision (this is for example the case for a perfect molecular gas);

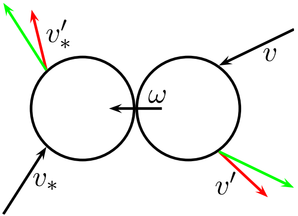
$\rightarrow$ inelastic when a fraction of the kinetic energy is dissipated during a collision (this is the case of a granular gas).

Inelastic Hard Spheres

- Microscopic dynamics: conservation of impulsion and dissipation of kinetic energy;
- Normal restitution coefficient $e \in [0, 1]$;
- Parametrization of the post-collisional velocities (v', v'_*) as a function of the pre-collisional velocities (v, v_*):

$$\begin{cases} v' = v - \dfrac{1+e}{2}\,((v - v_*) \cdot \omega)\,\omega, \\ \\ v'_* = v_* + \dfrac{1+e}{2}\,((v - v_*) \cdot \omega)\,\omega, \end{cases}$$

where $\omega \in \mathbb{S}^{d-1}$ is the impact direction;



(green is elastic, red is inelastic)

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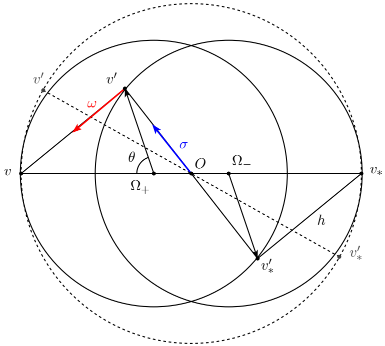

The Inelastic Collision Operator (weak form)

For two velocities $v, v_* \in \mathbb{R}^d$, we set $u := v - v_*$, $\hat{u} := \frac{u}{|u|}$; if ψ is a smooth test function, $\psi' := \psi(v')$, $\psi_* := \psi(v_*)$, $\psi'_* := \psi(v'_*)$.

Weak form of the collision operator (σ-representation)

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v)\,\psi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |u|\, f_*\, f\, \left(\psi' + \psi'_* - \psi - \psi_*\right) b(\hat{u}\cdot\sigma)\,d\sigma\,dv\,dv_*,$$

where b is the collisional cross section, which verifies

$$0 < \beta_1 \le b(x) \le \beta_2 < \infty, \quad \forall x \in [-1,1].$$

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$$0 < \beta_1 \leq b(x) \leq \beta_2 < \infty, \quad \forall x \in [-1,1].$$

Setting $\widetilde{b}(t) := 3|t|b(1-2t^2)$, one also has the ω-representation:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v)\,\psi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} |u|\, f_*\, f\, \left(\psi' + \psi'_* - \psi - \psi_*\right) \widetilde{b}(\hat{u})\,d\omega\,dv\,dv_*.$$

The Inelastic Collision Operator (strong form)

If e is non-zero, $(v, v_*, \sigma) \to (v', v'_*, \sigma)$ is not an involution, so one has to work a little bit more than in the elastic case for the strong form of the collision operator:

Strong form of the collision operator (σ-representation)

$$\mathcal{Q}_e(f,f)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \frac{|u|}{e} b_e^+(\hat{u}\cdot\sigma) \left({}'f\, {}'f_* - |u|\, f\, f_* \right) d\sigma\, dv_*,$$

with $b_e^+(s)$ given by

$$b_e^+(s) = b\left(\frac{(1+e^2)s - (1-e^2)}{(1+e^2) - (1-e^2)s}\right) \frac{\sqrt{2}}{\sqrt{(1+e^2) - (1-e^2)s}}. \tag{2}$$

In these expressions, the precollisional velocities are given in the σ-representation by

$$\begin{aligned} {}'v &= \frac{v+v_*}{2} + \frac{1-e}{4e}(v-v_*) + \frac{1+e}{4e}|v-v_*|\sigma, \\ {}'v_* &= \frac{v+v_*}{2} - \frac{1-e}{4e}(v-v_*) - \frac{1+e}{4e}|v-v_*|\sigma. \end{aligned} \tag{3}$$

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Macroscopic Properties of the Operator

Conservation of mass and momentum, dissipation of kinetic energy:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ -(1-e^2)D(f,f) \end{pmatrix}$$

where $D(f,f) \geq 0$ is the energy dissipation functional:

$$D(f,f) := b_1 \int_{\mathbb{R}^d \times \mathbb{R}^d} f\, f_* \, |v - v_*|^3 \, dv \, dv_*, \qquad b_1 \geq 0,$$

$$\geq b_1 \rho^{5/2} \left(\int_{\mathbb{R}} f(v) \, |v-u|^2 dv \right)^{\frac{3}{2}} \geq 0.$$

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Microscopic consequence: Concentration in the velocity space!

Some Consequences of the Energy Dissipation 1/2

If $e < 1$ (true inelasticity),

- Trivial equilibria (Dirac masses) because of the temperature dissipation: linearized study difficult;
- Particle velocities highly correlated;
- Mathematical analysis possible only in the L^1 Banach setting: no scalar products, perturbative theory intricate;
- No formal entropy dissipation structure[4]: very few energy estimates available, large time behavior difficult to investigate:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f,f)(v) \log f(v)\, dv = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_* \, f \left[\log\left(\frac{f' \, f'_*}{f \, f_*} \right) - \frac{f' \, f'_*}{f \, f_*} + 1 \right] B \, d\sigma \, d\!\!$$
$$+ \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \left(f'_* \, f' - f_* \, f \right) B \, d\sigma \, dv \, dv_*.$$

[4] Nevertheless, extensive numerical experiments show that the Boltzmann entropy would be a good candidate, see Garcia, Maynar, Mischler, Mouhot, TR, Trizac, *JSM* (2015)

Some Consequences of the Energy Dissipation 2/2

If $e < 1$ (true inelasticity),

- Macroscopic description formally given by the pressureless gas dynamics: creation of singularities in finite time;
- Creation of spatial d'inhomogeneities and zero temperature zones (clustering phenomenon).
- Meaningful even in dimension 1: the collision process is given by

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*) \right\}.$$

Monokinetic Distribution and Pressureless Dynamics

Define the first moments of f as

$$(\rho, \rho \boldsymbol{u}, \mathcal{E}, d\rho T) = \int_{\mathbb{R}^d} f(v)\varphi(v)dv, \qquad \varphi(v) = \left(1, v, |v|^2/2, |v - \boldsymbol{u}|^2\right).$$

Dissipation of energy formally yields that when $\varepsilon \to 0$,

$$f(t,x,v) \rightharpoonup_{\varepsilon \to 0} \rho(x)\delta_0\left(v - \boldsymbol{u}(x)\right), \quad \forall (x,v) \in \Omega \times \mathbb{R}^d,$$

where, using the macroscopic properties of $\mathcal{Q}_e$,

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \partial_t (\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) = \mathbf{0}. \end{cases}$$

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Pressureless Euler/Sticky particles equation: Generation of δ-singularity in finite time and transient clusters formation.

Quasi-Elastic Limit and Compressible Euler Dynamics

One can show[5] that if $1-e \sim \varepsilon$, then

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{\mathcal{Q}_1(f,f)}{\varepsilon} + \frac{1-e}{\varepsilon}\mathcal{I}(f,f) + \mathcal{O}\left(\frac{(1-e)^2}{\varepsilon}\right).$$

for an explicit friction operator $\mathcal{I}$.
When $\varepsilon \to 0$, there is a compressible Euler-like dynamics[6]

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \partial_t(\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho(\boldsymbol{u} \otimes \boldsymbol{u} + T\boldsymbol{I})) = \boldsymbol{0}, \\ \partial_t \mathcal{E} + \nabla_x \cdot (\boldsymbol{u}(\mathcal{E} + \rho T)) = -C_d \rho^2 T^{\frac{3}{2}}, \end{cases}$$

which also exhibits cluster formation[7].

[5] G. Toscani (2004)
[6] The closure is made thanks to the Maxwellian

$$\mathcal{M}_{\rho,\boldsymbol{u},T}(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{|v-\boldsymbol{u}|^2}{2T}\right).$$

[7] Carrillo-Salueña, 2006 and then Carrillo-Poëschel-Salueña 2009 for the Navier-Stokes case

Space homogeneous setting

Let us consider the space-independent equation $\partial_t f = \mathcal{Q}_e(f, f)$

- Cauchy problem for different types of kernels: Toscani (2000), Bobylev-Carrillo-Gamba (2000), Bobylev-Cercignani-Toscani (2003), Mischler-Mouhot-Ricard (2006);
- Qualitative behavior: Bobylev-Gamba-Panferov (2004), Gamba-Panferov-Villani (2004);
- Cooling process and asymptotic/self-similar behavior: Li-Toscani (2004), Mischler-Mouhot (2006), Alonso-Lods (2010, 2012), TR (2012);
- Stability and convergence towards the (unique) self-similar solution: Mischler-Mouhot (2009), Alonso-Lods (2011, 2013), Cañizo-Lods (2016), Alonso-Bagland-Cheng-Lods (2017);
- Entropy decay: No rigorous result but some study in that direction by Garcia-Maynar-Mischler-Mouhot-TR-Trizac (2015);
- Review papers: Villani (2006), Carrillo-Hu-Ma-TR (2021).

Space inhomogeneous setting

Full space-dependent equation $\partial_t f + v \cdot \nabla_x f = \mathcal{Q}_e(f, f)$

- Cauchy problem 1d: Benedetto-Caglioti-Pulvirenti (1997, 2001, 2002), with an idea due to Bony (1987);
- Cauchy problem near vacuum: Alonso (2009);
- Cauchy problem quasi-homogeneous, in the torus, thermal bath: Tristani (2013);
- Hydrodynamic limits:
  - Formal derivation: Toscani (2004), Carlen-Chow-Grigo (2010);
  - Spectral analysis of the linearized model: TR (2014);
  - Pressureless-Euler limit: Jabin-TR (2016);
  - Incompressible Navier-Stokes-Fourier with forcing: Alonso-Lods-Tristani (2020).

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Pressureless Euler system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \boldsymbol{u}) = 0, \\ \\ \partial_t (\rho\, \boldsymbol{u}) + \nabla_x \cdot (\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{0}. \end{cases}$$

Attract a lot of interest from the physics community because of its description of the large scale structure of the universe: Silk-Szalay-Zeldovich (1983)

- A rather delicate equation. Exhibits shocks because u (formally) solves Burgers equation where $\rho > 0 \Rightarrow$ concentration in $\rho \Rightarrow$ ill-posed system!
- Nevertheless, wellposedness theory exists by imposing semi-Lipschitz condition on u: Bouchut-James (1999), Boudin (2000), Huang-Wang (2001-2004);
- Model obtained as the hydrodynamic limit of the sticky particle model
  - In dimension 1: E-Rykov-Sinai (1996), Brenier-Grenier (1998),
  - In dimension 2: Berthon-Breuss-Titeux (2006), Chertock-Kurganov-Rykov (2007).

One Dimensional Sticky Particles

→ Known results. Hydrodynamic limit of the one dimensional N−particles system with sticky collisions[8] towards the pressureless Euler equations[9]

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0, \\ \partial_t(\rho\, u) + \partial_x\left(\rho\, u^2\right) = 0. \end{cases} \tag{4}$$

→ Method. Convergence of the empirical density towards a monokinetic distribution, using (among other arguments) the (microscopic) Oleinik property

$$\sup_{i\in\{1,\dots,N\}} \frac{(v_{i+1}-v_i)_+}{(x_{i+1}-x_i)_+} < \frac{1}{t},$$

where (x_i, v_i) represents the position and velocity of the i^{th} particle at time t.

[8] after collisions, the two particles stick together and travel with their average velocity.

[9] E-Rykov-Sinai, *CMP* (1996); Brenier-Grenier, *SINUM* (1998).

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Convergence of the particle distribution function $f(t, x, v)$ solution to the granular gases equation towards the monokinetic density

$$f(t, x, v) = \rho(t, x)\, \delta\left(x - u(t, x)\right),$$

where $(\rho,\, u)$ solution to the pressureless Euler system (4)?

[8]after collisions, the two particles stick together and travel with their average velocity.
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Hydrodynamic Limit of the Granular Gases Equation

Theorem (P.-E. Jabin, TR, QAM 2017)

Consider a sequence of weak solutions $f_\varepsilon(t,x,v) \in L^\infty([0,\,T],\; L^p(\mathbb{R}^2))$ *for some* $p > 2$ *and with total mass* 1 *to the granular gases Eq.* (1) *such that for any* k

$$\sup_\varepsilon \int_{\mathbb{R}^2} |v|^k \, f_\varepsilon^0(x,v)\,dx\,dv < \infty, \qquad \sup_\varepsilon \int_{\mathbb{R}^2} |x|^2 \, f_\varepsilon^0(x,v)\,dx\,dv < \infty,$$

and f_ε^0 *is, uniformly in* ε, *in some* L^p *for* $p > 1$

$$\sup_\varepsilon \int_{\mathbb{R}^2} (f_\varepsilon^0(x,v))^p \,dx\,dv < \infty. \tag{5}$$

Then any weak- limit* f *of* f_ε *is monokinetic,* $f = \rho(t,x)\,\delta(v - u(t,x))$ *for* $a.e.\ t$, *where* $\rho,\ u$ *are a solution in the sense of distributions to the pressureless system* (4) *while* u *has the Oleinik property for any* $t > 0$

$$u(t,x) - u(t,y) \le \frac{x-y}{t}, \quad \text{for } \rho \ a.e.\ x \ge y.$$

A global, dissipated functional?

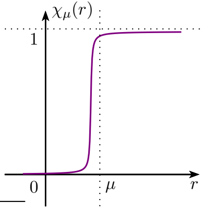
→ Setting. Assume that $f \in L^\infty(\mathbb{R}_+,\ M^1(\mathbb{R}^2))$ is a positive, bounded measure with compact support in $(x, v) \in [-R,\ R]^2$, and solves

(6) $$\partial_t f + v\,\partial_x f = -\partial_{vv} m/\varepsilon,$$

where m is an unknown positive measure in $M^1(\mathbb{R}_+ \times \mathbb{R}^2)$.

→ A "new" functional.[10] For any $t \geq 0$, $\eta > 0$, $k \geq 1$, $\mu > 0$, define

$$\mathcal{L}_{\eta,\mu,k}(f)(t) := \int \frac{(v-w)_+^{k+2}}{(x-y+\eta)^k} \chi_\mu(x-y)\, f(t,x,v)\, f(t,y,w)\, dv\, dw\, dx\, dy$$



[10] See also Bony (1987), Cercignani (1992), Biriuk, Craig, Panferov (2006)

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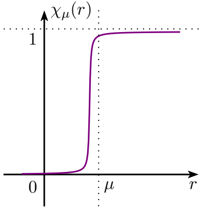
→ First observation. Let $(x_i(t), v_i(t))_{1\leq i\leq N}$ for $N \in \mathbb{N}$ a solution to the sticky particles system and f_N the associated empirical measure. Then

$$0 \leq \mathcal{L}_{\eta,\mu,k}(f_N)(T) \leq C_{N,k}.$$



[10] See also Bony (1987), Cercignani (1992), Biriuk, Craig, Panferov (2006)

A More General Result

Theorem (P.-E. Jabin, TR, QAM 2017)

Consider a sequence $f_\varepsilon \in L^\infty([0,\ T],\ M^1(\mathbb{R}^2))$ *of solutions to* (6) *with mass* 1 *for a corresponding sequence of non negative measures* m_ε*. Assume that for any* k

$$\sup_\varepsilon \sup_{t\in[0,\,T]} \int_{\mathbb{R}^2} |v|^k\, f_\varepsilon(t,dx,dv) < \infty, \quad \sup_\varepsilon \sup_{t\in[0,\,T]} \int_{\mathbb{R}^2} |x|^2\, f_\varepsilon(t,dx,dv) < \infty.$$

Assume moreover that f_ε *satisfies a (technical) trace condition on* t *and that*

$$\sup_\varepsilon \sup_{\eta,\mu} \mathcal{L}_{\eta,\mu,0}(f_\varepsilon)(t=0) < \infty.$$

Then any weak- limit* f *of* f_ε *solves the sticky particles dynamics in the sense that* $\rho = \int_{\mathbb{R}} f(t,x,dv)$ *and* $j = \int_{\mathbb{R}} v\, f(t,x,dv) = \rho\, u$ *are a distributional solution to the pressureless system* (4) *while* u *has the Oleinik property for any* $t > 0$

$$u(t,x) - u(t,y) \leq \frac{x-y}{t}, \quad \text{for } \rho \text{ a.e. } x \geq y.$$

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Spectral discretization of Boltzmann collision operator

- If the distribution function f has compact support on $\mathcal{B}_0(R)$, then $\mathrm{supp}(\mathcal{Q}_e(f,f)(\cdot)) \subset \mathcal{B}_0(\sqrt{2}R)$.
- Thus, in order to write a spectral approximation which avoids aliasing, it is sufficient that $f(v)$ is restricted to $[-T,T]^d$ with $T \geq (2+\sqrt{2})R$.

[11] where we took $T = \pi$ and hence $R = \lambda\pi$ with $\lambda = 2/(3+\sqrt{2})$ for easier notations

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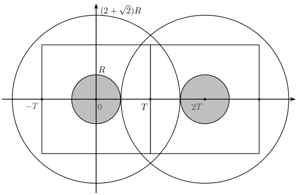
- Assuming $f(v) = 0$ on $[-T,T]^d \setminus \mathcal{B}_0(R)$, we extend $f(v)$ to a periodic function on the set $[-T,T]^d$.
- The choice $T = (3+\sqrt{2})R/2$ guarantees the absence of intersection between periods where f is different from zero.



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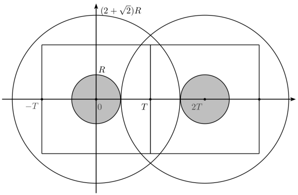
- The distribution function is represented as the truncated Fourier series[11]



$$f_N(v) = \sum_{k=-N}^{N} \hat{f}_k e^{ik\cdot v} \in \mathbb{P}_N,$$

$$\hat{f}_k = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(v) e^{-ik\cdot v}\, dv.$$

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Spectral discretization of Boltzmann collision operator II

- We then obtain a spectral quadrature by truncating the Boltzmann operator and projecting it onto the space of trigonometric polynomials of degree $\leq N$:

$$\hat{Q}_k = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathcal{Q}_e^R(f_N) e^{-ik\cdot v}\, dv, \quad k = -N, \dots, N.$$

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- By substituting the Fourier transform of f in $\hat{Q}$ one gets

$$\hat{Q}_k = \sum_{\substack{l,m=-\frac{N}{2} \\ l+m=k}}^{\frac{N}{2}-1} G(l,m) \hat{f}_l \hat{f}_m, \quad k = -N, \dots, N,$$

where the weight $G(l,m)$ is given by

$$G(l,m) = \int_{B_R} e^{-i\frac{\pi}{L} m\cdot g} \left[\int_{\mathbb{S}^{d-1}} B(|g|, \sigma\cdot\hat{g}) \left(e^{i\frac{\pi}{L}\frac{1+e}{4}(l+m)\cdot(g-|g|\sigma)} - 1 \right) d\sigma \right] dg.$$

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- The evaluation of $\hat{Q}$ requires $\mathcal{O}(N^{2d})$ operations.

Fast spectral discretization 1/2

Hu, Ma, 2018

To reduce the complexity the key idea is to render the weighted convolution into a pure convolution so that it can be computed efficiently by the FFT $\rightarrow$ low-rank approximation of $G(l,m)$:

$$G(l,m) \approx \sum_{p=1}^{N_p} \alpha_p(l+m)\beta_p(m),$$

where α_p and β_p will be determined later. Then

$$\hat{Q}_k \approx \sum_{p=1}^{N_p} \alpha_p(k) \sum_{\substack{l,m=-\frac{N}{2} \\ l+m=k}}^{\frac{N}{2}-1} \hat{f}_l \left(\beta_p(m)\hat{f}_m\right).$$

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Hence the total complexity to evaluate $\hat{Q}_k$ (for all k) is brought down to $\mathcal{O}(N_p N^d \log N)$, i.e., a few number of FFTs.

Fast spectral discretization 2/2

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We first split $G(l,m)$ into a gain term (not a convolution)

$$G_{\text{gain}}(l,m) := \int_{B_R} e^{-i\frac{\pi}{L}m\cdot g} \left[\int_{\mathbb{S}^{d-1}} B(|g|, \sigma\cdot\hat{g}) e^{i\frac{\pi}{L}\frac{1+e}{4}(l+m)\cdot(g-|g|\sigma)} d\sigma \right] dg,$$

and a loss term (already a convolution!) that one can precompute:

$$G_{\text{loss}}(m) := \int_{B_R} e^{-i\frac{\pi}{L}m\cdot g} \left[\int_{\mathbb{S}^{d-1}} B(|g|, \sigma\cdot\hat{g}) d\sigma \right] dg.$$

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For the gain term, we use a Gaussian quadrature to obtain

$$G_{\text{gain}}(l,m) \approx \sum_{\rho,\hat{g}} w_\rho w_{\hat{g}}\, \rho^{d-1} e^{-i\frac{\pi}{L}\rho\, m\cdot\hat{g}} F(l+m,\rho,\hat{g}),$$

where the function F is given by

$$F(l+m,\rho,\hat{g}) := \int_{\mathbb{S}^{d-1}} B(\rho, \sigma\cdot\hat{g}) e^{i\frac{\pi}{L}\rho\frac{1+e}{4}(l+m)\cdot(\hat{g}-\sigma)} d\sigma.$$

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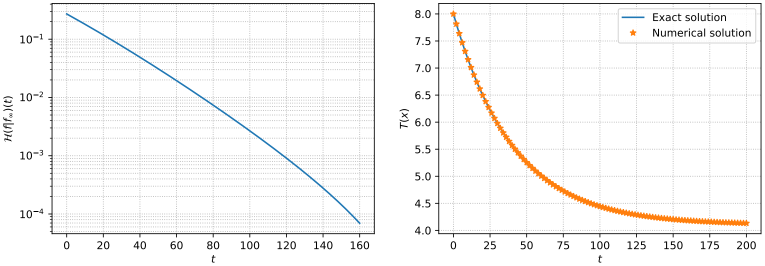
The total complexity to evaluate $\hat{Q}_k$ is $\mathcal{O}(MN^{d+1}\log N)$, where M is the number of points used on $\mathbb{S}^{d-1}$ and $M \ll N^{d-1}$

Trends to equilibrium (diffusively excited granular gas)

$\partial_t f = \mathcal{Q}_e(f,f) + \tau \Delta_v f$



Initial data is a centered reduced Maxwellian, $e = 0.95$ with heat bath $\tau = 0.05$.
Left: Semi-log plot of the relative entropy of f and $f_\infty = f(t = 100, v)$.
Right: numerical temperature (orange dots) and exact temperature (blue line).
Numerical parameters. $N_v^2 = 64 \times 64$, $N_\rho = 32$, $M_{\hat{g}} = 16$, $R = 20$, $L = 5(3+\sqrt{2})$ and $\Delta t = 0.01$.

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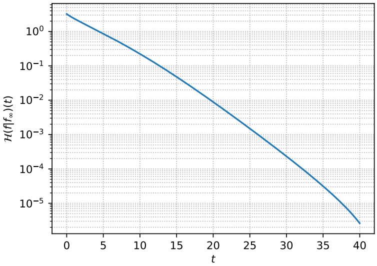

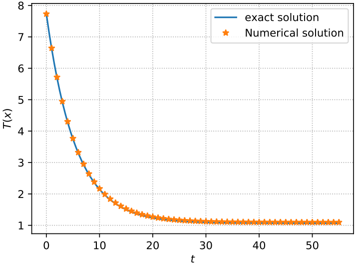

Initial data is an indicator function, $e = 0.5$ with heat bath $\tau = 0.1$.
Left: Semi-log plot of the relative entropy of f and $f_\infty = f(t = 100, v)$.
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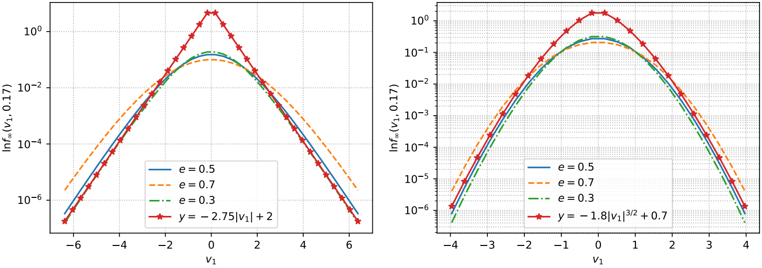
Tail behavior

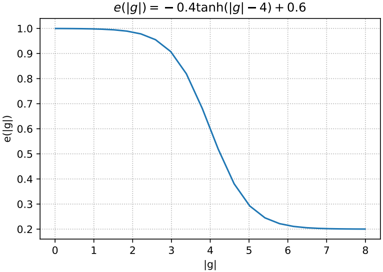
$\partial_t f = \mathcal{Q}_e(f,f) + \tau \Delta_v f$

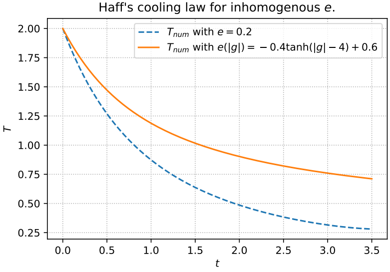


Equilibrium profile of $e = 0.3, 0.5, 0.7$ with heat bath $\tau = 0.1$.
Initial data is an indicator function.
Left: Semi-log plot of $f_\infty(v_1, 0.17) = f(t = 55, v_1, 0.17)$ for Maxwell molecules.
Right: Semi-log plot of $f_\infty(v_1, 0.17) = f(t = 55, v_1, 0.17)$ for hard spheres.
The red lines are the reference profiles.
Numerical parameters. $N_v^2 = 128 \times 128$, $N_\rho = 32$, $M_{\hat{g}} = 16$, $R = 20$, $L = 5(3 + \sqrt{2})$ and $\Delta t = 0.01$.

Haff's cooling law

$$\partial_t f = \mathcal{Q}_e(f,f)$$





Initial data is a shifted Maxwellian.
Left: plot of inhomogeneous e.
Right: comparison of temperature between constant $e = 0.2$ (dash line) and $e(|g| = \rho) = -0.4\tanh(\rho - 4) + 0.6$.
Numerical parameters. $N_v^3 = 32 \times 32 \times 32$, $N_\rho = 30$, $M_{\hat{g}} = 32$, $R = 8$, $L = 5(3+\sqrt{2})$ and $\Delta t = 0.01$.

Open problems

- AP scheme for the presureless Euler limit?
- AP scheme for the quasi-elastic limit?
- Rigorous multi-D hydrodynamic limit?

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Thank you for your attention!